1. (10 points) A function $f(x, y)$ is defined by $f(x, y) = \dfrac{x^3}{x^2 + y^2}$ at $(x, y) \neq 0$ and $f(0, 0) = 0$. Prove that $f(x, y)$ is continuous at $(0, 0)$.

Proof. Given $\epsilon > 0$, if $\|(x, y)\| < \epsilon$ then for $(x, y) \neq (0, 0)$

$$|f(x, y) - f(0, 0)| = \frac{|x|^3}{x^2 + y^2} \leq \frac{(x^2 + y^2)^{\frac{3}{2}}}{x^2 + y^2} = \sqrt{x^2 + y^2} = \|(x, y)\| < \epsilon.$$

Namely, $f(x, y)$ is continuous at $(0, 0)$. □

2. (10 points) Let $T_3(x)$ be the third order Taylor polynomial of

$$f(x) = \frac{x^2}{5-x}$$

centered at 0. Prove that $|f(x) - T_3(x)| < .01$ for $|x| < 1$.

If you use the remainder term (11) in pp.235, then you may obtain

$$|f(x) - T_3(x)| < \frac{25}{4^5} = \frac{25}{1024} < .0245.$$

In this case, you can get at most 8 points.

Proof. We have

$$f(x) = \frac{x^2}{5-x} = \frac{25 - (25 - x^2)}{5-x} = \frac{25}{5-x} - (x+5).$$

Hence,

$$f'(x) = \frac{25}{(5-x)^2} - 1, \qquad f''(x) = \frac{25 \cdot 2!}{(5-x)^3}, \qquad f^{(3)}(x) = \frac{25 \cdot 3!}{(5-x)^4}.$$

Therefore,

$$T_3(x) = \frac{x^2}{5} + \frac{x^3}{25}.$$

Hence,

$$\begin{aligned} f(x) - T_3(x) =& \frac{x^2}{5-x} - \frac{x^2}{5} - \frac{x^3}{25} = \frac{5x^2 - x^2(5-x)}{5(5-x)} - \frac{x^3}{25} \\ =& \frac{x^3}{5(5-x)} - \frac{x^3}{25} = \frac{5x^3 - x^3(5-x)}{25(5-x)} = \frac{x^4}{25(5-x)}. \end{aligned}$$

So, for $|x| < 1$ the following holds

$$|f(x) - T_3(x)| = \frac{|x|^4}{25(5-x)} < \frac{1}{25(5-x)} < \frac{1}{25(5-1)} = \frac{1}{100}.$$

□

3. (15 points) Prove that $\displaystyle\int_{0^+}^{\infty} \frac{(\sin x)^2}{x^2} dx$ converges.

(Fact: $\sin x$ is continuous and $\displaystyle\lim_{x \to 0^+} \frac{\sin x}{x} = 1$.)

Proof. The condition $\displaystyle\lim_{x \to 0^+} \frac{\sin x}{x} = 1$ implies that there exists $\delta > 0$ such that $\left| \frac{\sin x}{x} - 1 \right| < 1$ holds for $x \in (0, \delta)$. Namely, $\left| \frac{\sin x}{x} \right| < 2$ for $x \in (0, \delta)$. Since $\int_{0^+}^{\delta} 4 dx$ converges to 4δ, by the comparison theorem $\displaystyle\int_{0^+}^{\delta} \frac{(\sin x)^2}{x^2} dx$ converges.

Next, we can observe $\dfrac{(\sin x)^2}{x^2} \leq \dfrac{1}{x^2}$. Moreover, we have

$$\lim_{t \to +\infty} \int_{\delta}^{t} \frac{1}{x^2} dx = \lim_{t \to +\infty} \frac{-1}{x} \bigg|_{\delta}^{t} = \lim_{t \to +\infty} -\frac{1}{t} + \frac{1}{\delta} = \frac{1}{\delta},$$

namely $\displaystyle\int_{\delta}^{\infty} \frac{1}{x^2} dx$ converges. Hence, the comparison theorem implies the convergence of $\displaystyle\int_{\delta}^{\infty} \frac{(\sin x)^2}{x^2} dx$. Hence, $\displaystyle\int_{0^+}^{\infty} \frac{(\sin x)^2}{x^2} dx$ converges.

□

4. (25 points) Prove that $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2x^2 + x}$ is continuous but not uniformly continuous on $(0, 1]$.

Proof. First of all, the functions $u_n(x) = (n^2x^2 + x)^{-1}$ are continuous on $(0, 1]$ by Theorem 11.4C.

Given an interval $[r, 1] \subset (0, 1]$, we have

$$|u_n(x)| \leq \frac{1}{n^2x^2 + x} < \frac{1}{n^2x^2} \leq \frac{1}{n^2r^2}.$$

Moreover, $\sum_{n=1}^{\infty} \frac{1}{n^2r^2}$ converges, because $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Hence, Theorem 22.2B guarantees the uniform convergence of $\sum u_n(x)$ on $[r, 1]$. Therefore, $f(x)$ is continuous by Theorem 22.3.

Now, we assume that $f(x)$ is uniformly continuous on $(0, 1]$. Then, $\{f(a_n)\}$ is a Cauchy sequence if $\{a_n\} \subset (0, 1]$ is a Cauchy sequence. (See the practice pset or pset 5.) Since $\{\frac{1}{n}\}$ converges to 0, it is a Cauchy sequence. Therefore, $\{f(\frac{1}{n})\}$ is a Cauchy sequence, and thus it converges.

However,

$$\sum_{k=1}^{n} u_k\left(\frac{1}{n}\right) = \sum_{k=1}^{n} \left(\frac{k^2}{n^2} + \frac{1}{n}\right)^{-1} \geq \sum_{k=1}^{n} \left(\frac{n^2}{n^2} + \frac{1}{n}\right)^{-1} = \frac{n}{1 + \frac{1}{n}}.$$

Moreover, $u_i(\frac{1}{n}) > 0$ implies

$$\frac{n}{1 + n^{-1}} \leq \sum_{k=1}^{n} u_k\left(\frac{1}{n}\right) \leq \sum_{k=1}^{m} u_k\left(\frac{1}{n}\right),$$

for all $m \geq n$. Thus, the limit location theorem leads to $\frac{n}{1+n^{-1}} \leq f(\frac{1}{n})$. Namely, $f(\frac{1}{n})$ tends to the infinity, and thus diverges. □

5. (20 points) We define a function f on $\mathbb{R}^2$ by $f(x,y) = x\sin\Big(-\dfrac{1}{x^2+y^2}\Big)$ at $(x,y) \neq (0,0)$, and $f(0,0)=0$. Prove that following.
(a) $f(x,y)$ is continuous at $(0,0)$. (Thus, it is continuous on $\mathbb{R}^2$.)
(b) $f(x,y)$ is uniformly continuous on $\mathbb{R}^2$.

(You can use the fact that $f(x,y)$ is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$. Also, you can use the fact that $\sin t$ is differentiable and $(\sin t)' = \cos t$.)

Basically, it is the same to the problems 8 in the practice pset. Reading the proof, it would be good to draw the regions S_1, S_2 and the lines of the integrals.

Proof. Given $\epsilon > 0$, if $(x,y) \neq (0,0)$ and $\|(x,y)\| < \epsilon$ then

$$|f(x,y) - f(0,0)| \leq |x| \leq \sqrt{x^2+y^2} = \|(x,y)\| < \epsilon,$$

namely $f(x,y)$ is continuous at $(0,0)$. Thus, $f(x,y)$ is continuous on $\mathbb{R}^2$.

Next, we define $S_1 = [-2,2]^2 = \{(x,y) : |x| \leq 2, |y| \leq 2\}$. Then, S_1 is a compact set by Theorem 24.6. Therefore, $f(x,y)$ is uniformly continuous on S_1 by Theorem 24.7C.

We also define $S_2 = \mathbb{R}^2 \setminus [-1,1]^2 = \{(x,y) : \max(|x|,|y|) \geq 1\}$. Given $x \in \mathbb{R}$, we define $g_x(y) = f(x,y)$. Then, at $(x,y) \in S_2$ we have

$$|g_x'(y)| = \left|\frac{2xy}{(x^2+y^2)^2}\cos\Big(-\frac{1}{x^2+y^2}\Big)\right| \leq \frac{2xy}{(x^2+y^2)^2} \leq \frac{1}{x^2+y^2} \leq \frac{1}{\max(x^2,y^2)} \leq 1.$$

Similarly, we define $h_y(x) = f(x,y)$. Then at $(x,y) \in S_2$ we have

$$\begin{aligned} |h_y'(x)| &\leq \left|\sin\Big(-\frac{1}{x^2+y^2}\Big)\right| + \left|\frac{2x^2}{(x^2+y^2)^2}\cos\Big(-\frac{1}{x^2+y^2}\Big)\right| \\ &\leq 1 + \frac{2}{x^2+y^2} \leq 1 + \frac{2}{\max(x^2,y^2)} \leq 3. \end{aligned}$$

Given $\epsilon > 0$, if $(x_1,y_1),(x_2,y_2) \in S_2$ and $\|(x_1,y_1)-(x_2,y_2)\| < \frac{1}{10}\min(\epsilon,1)$, then we have $|x_1-x_2|, |y_1-y_2| < \frac{1}{10}\min(\epsilon,1)$.

We may assume $|x_1| \geq |y_1|$ and $|x_2| \geq |y_2|$. Then, we have $x_1, x_2 \geq 1$ because of $|x_1 - x_2| \leq \frac{1}{10}$ and $|x_1|, |x_2| \geq 1$. Thus, the FTC implies

$$\begin{aligned} &|f(x_1,y_1) - f(x_1,y_2)| = |g_{x_1}(y_1) - g_{x_1}(y_2)| = \Big|\int_{y_1}^{y_2} g_{x_1}'(y)dy\Big| \\ &\leq \int_{\min(y_1,y_2)}^{\max(y_1,y_2)} |g_{x_1}'(y)|dy \leq \int_{\min(y_1,y_2)}^{\max(y_1,y_2)} dy = |y_1-y_2| < \frac{\epsilon}{4}, \end{aligned}$$

because we have $(x_1, y) \in S_2$ for $y \in \mathbb{R}$ by $x_1 \geq 1$. Also,

$$|f(x_1, y_2) - f(x_2, y_2)| = |h_{y_2}(x_1) - h_{y_2}(x_2)| = \Big| \int_{x_1}^{x_2} h'_{y_1}(x)dx \Big|$$

$$\leq \int_{\min(x_1,x_2)}^{\max(x_1,x_2)} |h'_{y_1}(x)|dx \leq \int_{\min(x_1,x_2)}^{\max(x_1,x_2)} 3dy = 3|x_1 - x_2| < \frac{3\epsilon}{4},$$

namely $|f(x_1, y_1) - f(x_2, y_2)| < \epsilon$.

In the case $|y_1| \geq |x_1|$ and $|y_2| \geq |x_2|$, one can obtain $|f(x_1, y_1) - f(x_2, y_2)| < \epsilon$ in the same manner.

In the case $|x_1| \geq |y_1|$ and $|y_2| \geq |x_2|$, one can obtain

$$|f(x_1, y_1) - f(x_2, y_2)| \leq |g_{x_1}(y_1) - g_{x_1}(y_2)| + |h_{y_2}(x_1) - h_{y_2}(x_2)| \leq \epsilon.$$

In the case $|y_1| \geq |x_1|$ and $|x_2| \geq |y_2|$, one can obtain

$$|f(x_1, y_1) - f(x_2, y_2)| \leq |h_{y_1}(x_1) - h_{y_1}(x_2)| + |g_{x_2}(y_1) - g_{x_2}(y_2)| \leq \epsilon.$$

Hence, $f(x, y)$ is uniformly continuous on S_2.

In conclusion, given $\epsilon > 0$ there exist $\delta_1, \delta_2 > 0$ such that $|f(x_1, y_1) - f(x_2, y_2)| < \epsilon$ holds if $\|(x_1, y_1) - (x_2, y_2)\| < \delta_i$ and $(x_1, y_1), (x_2, y_2) \in S_i$ for each $i = 1, 2$. We now define $\delta = \min(\delta_1, \delta_2, \frac{1}{10})$ and assume $\|(x_1, y_1) - (x_2, y_2)\| < \delta$. Then, we have $(x_1, y_1), (x_2, y_2) \in S_1$ or $(x_1, y_1), (x_2, y_2) \in S_2$. Thus, $|f(x_1, y_1) - f(x_2, y_2)| < \epsilon$, namely $f(x, y)$ is uniformly continuous.

□

When you study the Cauchy integral in Complex Analysis, it would be good to compare with this problem.

The Cauchy integral is much more elegant.

6. (20 points) Let $f(x,y)$ be a continuous function on $\mathbb{R}^2$ such that

$$\lim_{\|(x,y)\|\to+\infty} f(x,y) = \frac{x}{\sqrt{x^2+y^2}}.$$

Prove that $f(x,y)$ is bounded on $\mathbb{R}^2$.

Proof. There exists $R > 0$ such that

$$\left| f(x,y) - \frac{x}{\sqrt{x^2+y^2}} \right| \le 1,$$

holds for $\|(x,y)\| > R$. Namely,

$$|f(x,y)| \le \left| f(x,y) - \frac{x}{\sqrt{x^2+y^2}} \right| + \left| \frac{x}{\sqrt{x^2+y^2}} \right| \le 2,$$

holds for $\|(x,y)\| > R$.

Next, $K = \{(x,y) : x^2 + y^2 \le R^2\}$ is a compact set by Theorem 25.1B. Thus, $f(x,y)$ attains its maximum M_1 and minimum M_2 on K by Theorem 24.7B. Hence, we have $M_2 \le f(x,y) \le M_1$ on K, namely $|f(x,y)| \le |M_1| + |M_2|$ on K.

In conclusion,

$$|f(x,y)| \le |M_1| + |M_2| + 2,$$

holds for $(x,y) \in \mathbb{R}^2$. $\square$

7. (20 points) Determine whether the following statements are true or false. If false then provide a counterexample. You do not need to verify your answer.

(1) A infinitely many times differentiable function is a analytic function.

Proof. F: $f(x) = e^{-\frac{1}{x}}$ at $x > 0$ and $f(x) = 0$ for $x \leq 0$. Then, its Taylor series at 0 is $0 \neq f(x)$. □

(2) Let $\{U_n\}_{n\in\mathbb{N}}$ be a sequence of open sets in $\mathbb{R}^2$. Then, the intersection $\cap_{n=1}^{\infty} U_n$ is an open set in $\mathbb{R}^2$.

Proof. F: $U_n = \{(x, y) : x^2 + y^2 < \frac{1}{n^2}\}$ are open sets, but $\cap_{n=1}^{\infty} U_n = \{(0, 0)\}$ is a closed set. □

(3) Let $f(x, y)$ be a continuous function defined on $\mathbb{R}^2$. Then, $f(x, y)$ has the maximum on the set $S = \{(x, y) : x^2 \leq 1, -1 \leq x + y \leq 1\}$.

Proof. T: You can draw the set S and simply check that S is contained the unit disk. Hence, S is bounded. Also $A = \{x^2 \leq 1\}$, $B = \{x + y \geq 1\}$, and $C = \{x + y \leq 1\}$ are closed by Theorem 25.1B. Thus, $S = A \cap B \cap C$ is closed. Thus, Theorem 25.2 implies that S is compact. So, $f(x, y)$ has the maximum on S by Theorem 24.7B. □

(4) Let $f(x, y)$ be a continuous function on $\mathbb{R}^2$ such that

$$\lim_{\|(x,y)\|\to+\infty} f(x, y) = \frac{x}{\sqrt{x^2 + y^2}}.$$

Then, $f(x)$ attains its maximum or minimum on $\mathbb{R}^2$.

Proof. F: $f(x) = \frac{2x}{\pi\sqrt{x^2+y^2}} \arctan(x^2 + y^2)$ at $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$.

If you don't like arctan, you can replace it by any continuous function $g(x^2 + y^2)$ such that $|g(r)| < 1$, $g(0) = 0$, and $g(r) \to 1$ as $r \to +\infty$. □

(5) Assume that $f(x)$ is a uniformly continuous function defined on $[1, +\infty)$. Then, $\dfrac{f(x)}{x}$ is bounded on $[1, +\infty)$.

Proof. T: There exists $\delta > 0$ such that $|f(x) - f(y)| < 1$ if $|x - y| \le \delta$ and $x, y \ge 1$. Then, given $x \ge 1$, there exists a natural number m such that $1 + (m-1)\delta \le x < 1 + m\delta$. So, we have

$$\begin{aligned}|f(x) - f(1)| \le& |f(x) - f(1 + (m-1)\delta)| + \sum_{k=1}^{m-1} |f(1 + k\delta) - f(1)| \\ \le& 1 + \sum_{k=1}^{m-1} 1 = m.\end{aligned}$$

Thus, $x < 1 + m\delta$ yields

$$|f(x)| \le |f(1)| + |f(x) - f(1)| \le |f(1)| + m < |f(1)| + \frac{1}{\delta}(x-1) < |f(1)| + \frac{x}{\delta}.$$

Hence, for $x \ge 1$

$$\frac{|f(x)|}{x} \le \frac{|f(1)|}{x} + \frac{1}{\delta} \le |f(1)| + \frac{1}{\delta}.$$

□

Remind that differentiable functions with bounded derivatives are uniformly continuous. But the uniform continuity does not imply the boundedness of the derivative. However, the boundedness of $\frac{f(x)}{x}$ *implies that the slope of* $f(x)$ *is bounded in "large scale".*

(6) If $\int_0^1 f(x)dx$ and $\int_0^1 g(x)dx$ converge, then $\int_0^1 f(x)g(x)dx$ converges.

Proof. F: $f(x) = g(x) = \frac{1}{\sqrt{x}}$. □

8. (20 points, bonus problem) Let $\Gamma = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ denote the unit circle.

We call a subset $U \subset \Gamma$ relatively open if there exists an open set $\bar{U} \subset \mathbb{R}^2$ such that $U = \Gamma \cap \bar{U}$. Also, a subset $V \subset \Gamma$ is relatively closed if there exists a closed set $\bar{V} \subset \mathbb{R}^2$ such that $V = \Gamma \cap \bar{V}$.

Prove that if a non-empty subset $A \subset \Gamma$ is relatively open and closed, then $A = \Gamma$.

Proof. To begin with, we observe that $\Gamma = \{(\cos\theta, \sin\theta) : \theta \in \mathbb{R}\}$. We define a set $\Theta = \{\theta \in \mathbb{R} : (\cos\theta, \sin\theta) \in A\}$. Since A is a non-empty set, we may assume $0 \in \Theta$ without loss of generality.

Next, by definition of the relatively open sets, there exists an open set $\bar{U} \in \mathbb{R}^2$ such that $A = \bar{U} \cap \Gamma$. Thus, $(1,0) = (\cos 0, \sin 0) \in A \subset \bar{U}$. Hence, by definition of the open sets in $\mathbb{R}^2$, there exists $\epsilon > 0$ such that the open ball $B_\epsilon((1,0))$ is contained in $\bar{U}$. Therefore, $B_\epsilon((1,0)) \cap \Gamma = \bar{U} \cap \Gamma = A$.

On the other hand,

$$\begin{aligned} \|(1,0) - (\cos\theta, \sin\theta)\| =& \sqrt{(1-\cos\theta)^2 + (\sin\theta)^2} \\ =& \sqrt{2 - 2\cos\theta} = 2|\sin(\theta/2)|. \end{aligned}$$

Since $\lim_{\theta\to 0} 2\sin(\theta/2) = 0$, there exists $\delta > 0$ such that $2|\sin(\delta/2)| < \epsilon$ holds for $|\theta| \leq \delta$. Namely, $(\cos\theta, \sin\theta) \in B_\epsilon((1,0))$ holds for $\theta \in [0,\delta]$. Therefore, we have $[0,\delta] \subset \Theta$.

Now, we can define a set $S = \{s \in [0,+\infty) : [0,s] \subset \Theta\}$. Then, it is a non-empty set because of $\delta \in S$. If there exists some $s \in S$ with $s \geq 2\pi$, then we have $[0,2\pi] \subset [0,s] \subset \Theta$. Namely, we have the desired result $A = \Gamma$.

Thus, we may assume that $s < 2\pi$ holds for all $s \in S$, namely 2π is an upper bound for S. By the completeness theorem, $\bar{s} = \sup S$ exists.

If $\bar{s} \in S$ then $\bar{s} \in [0,\bar{s}] \subset \Theta$. Thus, $(\cos\bar{s}, \sin\bar{s}) \in A$. Then, there exists some $\delta' > 0$ such that $[\bar{s}, \bar{s}+\delta'] \subset \Theta$ as like the previous argument. Then, we have $[0,\bar{s}+\delta] = [0,\bar{s}] \cup [\bar{s},\bar{s}+\delta'] \subset \Theta$, namely $\bar{s}+\delta' \in S$. Contradiction.

We assume that $\bar{s} \notin S$. By the problem 9 in the pset 2, there exists a sequence $\{s_n\} \subset S$ such that $\lim s_n = s$. Since $\bar{s} \notin S$, we have $s_n \neq s$.

Now, we let $\bar{V}$ be the closed set in $\mathbb{R}^2$ with $A = \bar{V} \cap \Gamma$. Then,

$$(\sin s_n, \cos s_n) \in A \subset \bar{V}, \qquad \lim_{n\to+\infty} (\sin s_n, \cos s_n) = (\sin\bar{s}, \cos\bar{s}).$$

Thus, $(\sin\bar{s}, \cos\bar{s})$ is a cluster point of the closed set $\bar{V}$, and $(\sin\bar{s}, \cos\bar{s}) \in \bar{V} \cap \Gamma = A$. So, $\bar{s} \in \Theta$. However, we have

$$[0,\bar{s}) = \cup_{n=1}^{\infty} [0,s_n] \subset \Theta.$$

Thus, $[0,\bar{s}] = \{\bar{s}\} \cup [0,\bar{s}) \subset \Theta$, namely $\bar{s} \in S$. Contradiction.

$\square$